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EXACT SOLUTION OF THE THREE-DIMENSIONAL PROBLEM OF IDEAL PLASTICITY

S. I. Senashov

UDC 539.374

Let $r\theta z$ be a cylindrical coordinate system, σ_r, σ_θ, σ_z, $\tau_{r\theta}$, τ_{rz}, $\tau_{\theta z}$ be the components of the stress tensor, u, v, and w be the components of the velocity vector, and k be the yield stress under pure shear.

The equations of ideal plasticity with the von Mises yield condition are of the form

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0,$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{\theta z}}{\partial z}+\frac{2\tau_{r\theta}}{r}=0,$$

$$\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0,$$

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6\left(\tau_{r\theta}^2+\tau_{rz}^2+\tau_{\theta z}^2\right)=6k^2, \tag{1}$$

$$\lambda\frac{\partial u}{\partial r}=2\sigma_r-\sigma_\theta-\sigma_z,\quad \lambda\left(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)=2\sigma_\theta-\sigma_z-\sigma_r,$$

$$\lambda\frac{\partial w}{\partial z}=2\sigma_z-\sigma_r-\sigma_\theta,\quad \lambda\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z}\right)=2\tau_{\theta z},$$

$$\lambda\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=2\tau_{rz},\quad \lambda\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{1}{r}\frac{\partial u}{\partial\theta}\right)=2\tau_{r\theta},$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial\theta}+r\frac{\partial w}{\partial z}=0,$$

$$\sigma_r+\sigma_\theta+\sigma_z=3p.$$

We shall assume that

$$\tau_{rz}=\tau_{r\theta}=0. \tag{2}$$

We shall seek the solution of Eqs. (1) in the form

$$u=u^*(r)\operatorname{sh}\xi,\ v=v^*(r)\operatorname{ch}\xi,\ w=w^*(r)\operatorname{ch}\xi,\ p=p(r),\ \xi=z+\beta\theta, \tag{3}$$

where u^*, v^*, w^* are functions only of r and β is an arbitrary constant. Then we obtain from the incompressibility condition and Eqs. (2) a system of ordinary differential equations for determination of the functions u^*, v^*, w^*:

$$u^*+\frac{dw^*}{dr}=0,\quad r\frac{d}{dr}\left(\frac{v^*}{r}\right)+\frac{\beta}{r}u^*=0,\quad \frac{d}{dr}(ru^*)+\beta v^*+rw^*=0, \tag{4}$$

and the equation

$$d\sigma_r/dr+(\sigma_r-\sigma_\theta)/r=0 \tag{5}$$

remains for the determination of the pressure p. The system of equations (4) reduces to the Bessel equation

$$r^2u^{*\prime\prime}+ru^{*\prime}-(r^2+\beta^2+1)u^*=0.$$

The solution of this equation is of the form

$$u^*=C_1I_\nu(r)+C_2K_\nu(r),\ \nu=\sqrt{\beta^2+1}, \tag{6}$$

where I_ν are the Bessel functions of imaginary argument, K_ν is the MacDonald function, and C_1 and C_2 are arbitrary constants. If one sets $C_2 = 0$ in (6), the velocity field is of the form

Krasnoyarsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 153-155, July-August, 1984. Original article submitted November 22, 1983.

 0021-8944/84/2504-0648$08.50

$$u = C_1 I_\nu(r)\operatorname{sh}\xi,$$
$$v = -\left[rC_1\beta\int I_\nu(r)\frac{dr}{r^2}\right]\operatorname{ch}\xi,\quad w = -C_1\operatorname{ch}\xi\int I_\nu(r)\,dr. \tag{7}$$

The components of the stress tensor are equal to

$$\sigma_r = C + \int\frac{F(\varphi-1)}{r}dr,\quad \tau_{\theta z} = \psi F,$$
$$\sigma_\theta = \sigma_r + (\varphi-1)F,\ \sigma_z = \sigma_r - (2+\varphi)F,\ F = \sqrt{2}k(1+\varphi^2+(1+\varphi)^2+$$
$$+(1/2)\psi^2)^{-1/2}, \tag{8}$$
$$\varphi = (u^* + \beta v^*)/ru^{*\prime},\ \psi = (\beta w^* + rv^*)/ru^{*\prime}.$$

One can use the solution (7) and (8) to describe the plastic flow of a cylinder ($0 < r \leqslant R$, $-l \leqslant z \leqslant l$), loaded at the ends by a stress distribution according to the law

$$\sigma_z = -(2+\varphi)F + \int_0^r\frac{F(\varphi-1)}{r}dr,$$

and by the torsional moment

$$M = 2\pi\int_0^R \tau_{\theta z}r^2dr.$$

Assuming the lateral surface to be free of stresses, we determine the constant C from the condition $\sigma_r(R) = 0$.

If $C_2 \neq 0$ in formula (6), one can use the solution constructed to describe the plastic flow of a tube acted on by tensile stresses, a torsional moment, and internal pressure.

If one sets $\beta = 0$ in formulas (8), the components of the stress tensor will coincide with those found in [1] for the case of axisymmetric strain.

One can also use the solution found to describe the flow of a plastically nonuniform medium; for this it is sufficient to set $k = K(r)$ in formulas (1) and (8).

LITERATURE CITED

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